

CUMULATIVE GRAVITATIONAL LENSING IN NEWTONIAN PERTURBATIONS OF FRIEDMAN-ROBERTSON-WALKER COSMOLOGIES

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ABSTRACT. It is a common assumption amongst astronomers that, in the determination of the distances of remote sources from their apparent brightness, the cumulative gravitational lensing due to the matter in all the galaxies is the same, on average, as if the matter were uniformly distributed throughout the cosmos. The validity of this assumption is considered here by way of general Newtonian perturbations of Friedman-Robertson-Walker (FRW) cosmologies. The analysis is carried out in synchronous gauge, with particular attention to an additional gauge condition that must be imposed. The mean correction to the apparent magnitude-redshift relation is obtained for an arbitrary mean density perturbation. In the case of a zero mean density perturbation, when the intergalactic matter has a dust equation of state, then there is indeed a zero mean first order correction to the apparent magnitude-redshift relation for all redshifts. Point particle and Swiss cheese models are considered as particular cases.

1. INTRODUCTION

The purpose of this paper is threefold: first to give a general analysis of synchronous gauge Newtonian perturbations of Friedman-Robertson-Walker (FRW) cosmologies; second to obtain a condition which identifies a suitable homogenised FRW cosmology for comparison purposes, and third to obtain the relevant formula for the correction to the apparent magnitude-redshift relation for weak gravitational lensing.

The classic paper of Lifschitz & Khalatnikov (1963) generated much interest in the study of perturbations of FRW cosmologies. Among the influential papers that followed were those of Hawking (1966) and Sachs & Wolfe (1967). A review that covers both classical and quantum aspects is Mukhanov et al. (1992). One of the main topics in the literature is the growth of small perturbations in the early universe and the formation of the galaxies. The present work, by contrast, is concerned with the description of the universe as it is today, with the galaxies providing an essentially time independent perturbation of an FRW background.

Although the use of a gauge invariant approach is frequently advocated, since it eliminates any need to specify and interpret gauge conditions, it is nonetheless more appropriate here to employ a synchronous gauge. This gauge is known to involve a certain ambiguity, as was pointed out by Mukhanov et al. (1992), and has given rise to difficulties of interpretation (Press & Vishniac 1980). It will therefore be necessary in the present work to identify an appropriate supplementary gauge condition which admits a natural interpretation. This will be done in §6.

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Gravitational lensing has been considered by many authors and with different techniques (see e.g. Schneider *et al.* 1992; Bertotti 1966; Dyer & Roeder 1972, 1973, 1974; Kantowski 1998). One such technique has been the use of stochastic numerical integration of the lensing equations (Holz & Wald 1998; Dyer & Oates 1988). The effects of weak gravitational lensing on the cosmic microwave background radiation have been considered by Seljak (1996) and by Sachs & Wolfe (1967). The optical properties of the Swiss cheese model of Einstein & Strauss (1945, 1946) have been considered by Dyer & Roeder (1974). Both the Swiss cheese model and the point particle model of Newman & McVittie (1982) will be considered here as applications of a general theory.

Newtonian perturbations of FRW cosmologies have previously been considered by McVittie (1931), Harrison (1967) and Newman & McVittie (1982). A central difficulty is that if one neglects terms involving the spatial curvature constant k , the total gravitational potential for a uniform distribution of particles will be everywhere infinite. What does not seem to be widely appreciated is that this problem disappears when proper account is taken of either a positive or negative k . This will be discussed in §8*b*.

Much of the analysis will be carried out in the linear approximation in the gravitational coupling constant $\kappa = 8\pi G/c^2$. Consequently only weak gravitational lensing will be described. In particular, shear is neglected because it has a second order effect on luminosity. Caustics are also neglected. These may both be significant restrictions. There are now well-known examples of astronomical images that show the effects of caustics, although there are as yet no firm estimates of the proportion of images for which caustics are involved. It has been claimed by Ellis *et al.* (1998) that light rays followed back from our present location in time and space meet a galaxy within a redshift $0 < z < 5$. If this is correct then strong gravitational lensing is likely to be significant for images with $z \geq 5$.

The question as to whether, in general, weak gravitational lensing gives the same apparent magnitude-redshift relation on average as a best-fit FRW model was considered by Weinberg (1976) who gave two supporting arguments. The first was based on an analysis of double image lensing but was only valid for $q \ll 1$, i.e. for $\Omega \ll 1$. The second, subsequently generalised by Peacock (1986), was essentially based on photon conservation (Schneider *et al.* 1992, pp.99–100), and was valid for all Ω . Although widely accepted in the literature, these arguments have been criticised by Ellis *et al.* (1988), the first on the grounds that generic lensing gives rise to three images, not two, and the second at least in part because it effectively assumes the result to be proved. The criticism of Weinberg's first argument is presumably based on the odd number theorem of Burke (1981) and McKenzie (1985), according to which a transparent gravitational lens can give rise only to an odd number of images. However this theorem is flawed since, as was pointed out by Gottlieb (1994), it attempts to use 3-dimensional topology in a 4-dimensional setting. Nonetheless the criticism of Weinberg's second argument appears to stand. The present paper therefore gives a new supporting argument that, in the weak field limit, and when averaged over large angular scales, the apparent magnitude-redshift relation does indeed agree with that of a best-fit FRW model.

2. GENERAL PERTURBATIONS OF FRW COSMOLOGIES

The FRW metric, in standard form, is

$$\overset{\circ}{\mathbf{g}} = -dt^2 + S^2(t)\overset{\circ}{\mathbf{h}} \quad (2.1)$$

where $\overset{\circ}{\mathbf{h}}$ is a t -independent 3-metric given by

$$\overset{\circ}{\mathbf{h}} = \left(1 + \frac{kr^2}{4}\right)^{-2} \sum_{i=1}^3 (dx^i)^2. \quad (2.2)$$

Here and henceforth, indices i, j, \dots run from 1 to 3, whilst indices a, b, \dots will run from 0 to 3. For the most part it will be possible to describe the level surfaces of t with respect to an arbitrary coordinate system $\mathbf{x} = \{x^1, x^2, x^3\}$. The space-time may then be conveniently described with respect to the coordinate system $\{x^a : a = 0, \dots, 3\}$ with $x^0 = t$.

Quantities associated with $\overset{\circ}{\mathbf{g}}$ will carry a superscript $^\circ$. Covariant differentiation with respect to $\overset{\circ}{\mathbf{g}}$ will be denoted by $\overset{\circ}{\cdot}$; e.g. $\overset{\circ}{g}_{ab;\overset{\circ}{c}} = 0$. The energy tensor of $\overset{\circ}{\mathbf{g}}$ is

$$\overset{\circ}{T}^{ab} = (\overset{\circ}{\rho} + \overset{\circ}{p})u^a u^b + \overset{\circ}{p}\overset{\circ}{g}^{ab} \quad (2.3)$$

where $\overset{\circ}{g}^{ab}$ is the inverse of $\overset{\circ}{g}_{ab}$, $u^a := -\overset{\circ}{g}^{ab}t_{;\overset{\circ}{b}}$ is the unit future-directed normal to the level surfaces of t ,

$$\kappa\overset{\circ}{\rho}(t) = \frac{3k}{S^2(t)} + \frac{\dot{S}^2(t)}{S^2(t)} - \Lambda \quad (2.4)$$

$$\kappa\overset{\circ}{p}(t) = -\frac{k}{S^2(t)} - \frac{\ddot{S}(t)}{S(t)} + \Lambda \quad (2.5)$$

give the density $\overset{\circ}{\rho}(t)$ and pressure $\overset{\circ}{p}(t)$ respectively, and Λ is the cosmological constant, which is not assumed to vanish. A superscript dot denotes differentiation with respect to t .

The Hubble and deceleration parameters for $\overset{\circ}{\mathbf{g}}$ are defined by

$$\overset{\circ}{H}(t) := \frac{\dot{S}(t)}{S(t)} \quad (2.6)$$

$$\overset{\circ}{q}(t) := -\frac{\ddot{S}(t)S(t)}{\dot{S}^2(t)} \quad (2.7)$$

respectively whilst the dimensionless density parameter is defined by

$$\overset{\circ}{\Omega}(t) := \frac{\kappa\overset{\circ}{\rho}(t)}{3\overset{\circ}{H}^2(t)}. \quad (2.8)$$

Note that by (2.4) and (2.5) one has

$$\frac{d}{dt}\overset{\circ}{H}(t) = -\overset{\circ}{H}^2(t)(1 + \overset{\circ}{q}(t)) = \frac{k}{S^2(t)} - \frac{1}{2}\kappa(\overset{\circ}{\rho} + \overset{\circ}{p}). \quad (2.9)$$

Covariant differentiation with respect to $\overset{\circ}{\mathbf{h}}$ in the level surfaces of t will be denoted by a subscript $\overset{\circ}{\parallel}$, e.g. $\overset{\circ}{h}_{ij;\overset{\circ}{\parallel}k} = 0$. Quantities associated with the 3-metric $\overset{\circ}{\mathbf{h}}$, apart from $\overset{\circ}{\mathbf{h}}$ itself will, in addition to the superscript $^\circ$, carry a prefix $^{(3)}$. Thus

the Laplacian associated with $\overset{\circ}{\mathbf{h}}$ will be denoted by $^{(3)}\overset{\circ}{\Delta}$. The Riemann and Ricci tensors and Ricci scalar of $\overset{\circ}{\mathbf{h}}$ are

$$^{(3)}\overset{\circ}{R}{}^i{}_{jkl} = 2k\delta^i{}_{[k}\overset{\circ}{h}_{l]j} \quad (2.10)$$

$$^{(3)}\overset{\circ}{R}_{ij} = 2k\overset{\circ}{h}_{ij} \quad (2.11)$$

$$^{(3)}\overset{\circ}{R} = 6k \quad (2.12)$$

respectively.

The calculations of the remainder of this section pertain to a perturbation of the FRW metric (2.1) of the particular form

$$\mathbf{g} = -dt^2 + S^2(t)\mathbf{h} \quad (2.13)$$

where

$$\mathbf{h} = \overset{\circ}{\mathbf{h}} + \delta\mathbf{h} \quad (2.14)$$

is a 3-metric intrinsic to the level surfaces of t , i.e. transverse to $\overset{\circ}{\mathbf{u}}$, and where $\delta\mathbf{h}$ may depend upon all four coordinates.

The scale factor $S(t)$, the cosmological constant Λ and the spatial curvature constant k will all be taken to be the same as for the unperturbed metric $\overset{\circ}{\mathbf{g}}$. Note that since $\overset{\circ}{\mathbf{h}}$ is independent of t one has $\partial_t h_{ij} = \partial_t \delta h_{ij}$. Covariant differentiation with respect to \mathbf{h} in the level surfaces of t will be denoted by \parallel , e.g. $h_{ij\parallel k} = 0$. Quantities associated with \mathbf{h} , other than \mathbf{h} itself, will carry a prefix $^{(3)}$. Thus the Laplacian associated with \mathbf{h} will be denoted by $^{(3)}\Delta$ and the Ricci tensor and Ricci scalar of \mathbf{h} will be denoted by $^{(3)}R_{ij}$ and $^{(3)}R$ respectively. Indices $i, j \dots$ on quantities associated with \mathbf{h} will be raised and lowered by \mathbf{h}^{-1} and \mathbf{h} respectively; indices $a, b \dots$ on quantities associated with \mathbf{g} will be raised and lowered by \mathbf{g}^{-1} and $\overset{\circ}{\mathbf{g}}$ respectively. Differences between quantities associated with \mathbf{g} and $\overset{\circ}{\mathbf{g}}$, or \mathbf{h} and $\overset{\circ}{\mathbf{h}}$, will be denoted by δ , e.g. $\delta^{(3)}R_{ij} := ^{(3)}R_{ij} - ^{(3)}\overset{\circ}{R}_{ij}$.

The vector field $u^a = -t^{;a}$ is irrotational and geodesic. The integral curves of \mathbf{u} are to be regarded as the world lines of a preferred family of observers, described henceforth as comoving. In particular the galaxies are presumed to be comoving. The coordinate t thus carries physical significance.

The second fundamental form of the level surfaces of t is given by

$$\chi_{ab} := u_{a;b} . \quad (2.15)$$

Clearly χ_{ab} is transverse to u^a and so defines a 3-tensor intrinsic to the level surfaces of t :

$$\chi_{ij} = \Gamma_{ij}^t = \frac{1}{2}\partial_t(S^2(t)h_{ij}) = \dot{S}(t)S(t)h_{ij} + \frac{1}{2}S^2(t)\partial_t\delta h_{ij} . \quad (2.16)$$

The Hubble and deceleration parameters of u^a with respect to g_{ab} are defined by

$$H := u^a{}_{;a} = \frac{1}{3}\chi^a{}_a = \overset{\circ}{H}(t) + \frac{1}{6}h^{ij}\partial_t\delta h_{ij} \quad (2.17)$$

$$q := \partial_t\left(\frac{1}{H}\right) - 1 . \quad (2.18)$$

Note that H and q are functions of all four coordinates. By (2.16) the Gauss equation for ${}^{(3)}R^i_{jkl}$ leads to

$$\begin{aligned} {}^{(3)}R &= 6k - S^2(t)\kappa(\overset{\circ}{\rho} - \overset{\circ}{p}) + 2S^2(t)(\kappa\delta T_{tt} - \frac{\dot{S}(t)}{S(t)}h^{ij}\partial_t\delta h_{ij}) \\ &\quad + \frac{1}{2}h^{i[m}h^{j]n}(\partial_t\delta h_{ij})(\partial_t\delta h_{mn}) . \end{aligned} \quad (2.19)$$

From (2.16), (2.17), (2.18) and (2.19) one has

$$\delta\chi_{ij} = \frac{1}{2}\partial_t(S^2(t)\delta h_{ij}) \quad (2.20)$$

$$\delta H = \frac{1}{6}h^{ij}\partial_t\delta h_{ij} \quad (2.21)$$

$$\delta q = -\frac{1}{\overset{\circ}{H}}\partial_t\left(\frac{\delta H}{H}\right) - (1 + \overset{\circ}{q})\frac{\delta H}{H} \quad (2.22)$$

$$\delta {}^{(3)}R = 2S^2(t)(\kappa\delta T_{tt} - \frac{\dot{S}(t)}{S(t)}h^{ij}\partial_t\delta h_{ij}) + \frac{1}{2}h^{i[m}h^{j]n}(\partial_t\delta h_{ij})(\partial_t\delta h_{mn}) . \quad (2.23)$$

The components of the Ricci tensor of g_{ab} are given by

$$R_{tt} = -3\frac{\ddot{S}(t)}{S(t)} - \frac{1}{2S^2(t)}\partial_t(S^2(t)h^{ij}\partial_t\delta h_{ij}) - \frac{1}{4}h^{ik}h^{jl}(\partial_t\delta h_{ij})(\partial_t\delta h_{kl}) \quad (2.24)$$

$$R_{it} = R_{ti} = h^{jk}(\partial_t\delta h_{k[i}\delta_{j]}) \quad (2.25)$$

$$\begin{aligned} R_{ij} &= (2k + 2\dot{S}^2(t) + S(t)\ddot{S}(t) + \frac{1}{2}\dot{S}(t)S(t)h^{km}\partial_t\delta h_{km})h_{ij} \\ &\quad + \delta {}^{(3)}R_{ij} - 2k\delta h_{ij} + \frac{1}{2S(t)}\partial_t(S^3(t)\partial_t\delta h_{ij}) \\ &\quad + \frac{1}{4}S^2(t)h^{km}(\partial_t\delta h_{km})(\partial_t\delta h_{ij}) - \frac{1}{2}S^2(t)h^{km}(\partial\delta h_{ik})(\partial_t\delta h_{jm}) , \end{aligned} \quad (2.26)$$

the last of these having been obtained by means of (2.11). One also has

$$\begin{aligned} R &= 6\left(\frac{k}{S^2(t)} + \frac{\dot{S}^2(t)}{S^2(t)} + \frac{\ddot{S}(t)}{S(t)}\right) + \frac{1}{S^2(t)}\delta {}^{(3)}R \\ &\quad + \frac{1}{S^4(t)}\partial_t(S^4(t)h^{ij}\partial_t\delta h_{ij}) + \frac{1}{2}h^{i(j}h^{k)m}(\partial_t\delta h_{km})(\partial_t\delta h_{ij}) \end{aligned} \quad (2.27)$$

by means of (2.11) and

$$\delta {}^{(3)}R = h^{ij}{}^{(3)}R_{ij} - \overset{\circ}{h}{}^{ij}{}^{(3)}\overset{\circ}{R}_{ij} = h^{ij}(\delta {}^{(3)}R_{ij} - 2k\delta h_{ij}) . \quad (2.28)$$

Hence the energy tensor \mathbf{T} of \mathbf{g} , as defined by Einstein's equations

$$\kappa T_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} , \quad (2.29)$$

has components

$$\kappa T_{tt} = \kappa \overset{\circ}{\rho} + \frac{1}{2S^2(t)} \delta^{(3)} R + \frac{\dot{S}(t)}{S(t)} h^{ij} \partial_t \delta h_{ij} + \frac{1}{4} (h^{ij} \partial_t \delta h_{ij})^2 \quad (2.30)$$

$$\kappa T_{it} = \kappa T_{ti} = (h^{jk} \partial_t \delta h_{k[i} \delta h_{j]}) \quad (2.31)$$

$$\begin{aligned} \kappa T_{ij} = & \{ S^2(t) \kappa \overset{\circ}{p} - \frac{1}{2S(t)} \partial_t (S^3(t) h^{km} \partial_t \delta h_{km}) \\ & - \frac{1}{4} S^2(t) h^{p(q} h^{k)m} (\partial_t \delta h_{km}) (\partial_t \delta h_{pq}) \} h_{ij} \\ & + \delta^{(3)} R_{ij} - 2k \delta h_{ij} - \frac{1}{2} h_{ij} \delta^{(3)} R + \frac{1}{2S(t)} \partial_t (S^3(t) \partial_t \delta h_{ij}) \\ & + \frac{1}{4} S^2(t) h^{km} \{ (\partial_t \delta h_{km}) (\partial_t \delta h_{ij}) - 2(\partial_t \delta h_{ik}) (\partial_t \delta h_{jm}) \} \end{aligned} \quad (2.32)$$

The Weyl tensor $C^a{}_{bcd}$ of g_{ab} may be decomposed in the standard form

$$C_{abcd} = 8u_{[a} E_{b][d} u_{c]} + 2g_{a[c} E_{d]b} - 2g_{b[c} E_{d]a} - 2u^e \eta_{eabf} u_{[c} B_{d]}{}^f - 2u^e \eta_{ecd} u_{[a} B_{b]}{}^f \quad (2.33)$$

where η_{abcd} is the alternating tensor,

$$E_{ab} := C_{acbd} u^c u^d \quad (2.34)$$

$$B_{ab} := {}^* C_{acbd} u^c u^d \quad (2.35)$$

are the electric and magnetic parts of $C^a{}_{bcd}$, and

$${}^* C_{abcd} := \frac{1}{2} \eta_{abef} C^{ef}{}_{cd} = \frac{1}{2} \eta_{cdef} C^{ef}{}_{ab} \quad (2.36)$$

is the dual of $C^a{}_{bcd}$.

In order to compute E_{ab} one may use the identity

$$2u^b u_{a;[bc]} = R_{dabc} u^b u^d \quad (2.37)$$

to obtain

$$E_{ij} = \frac{1}{2} R_{ij} - \frac{1}{2} S^2(t) (R_{tt} + \frac{1}{3} R) h_{ij} - \frac{D}{\partial t} \chi_{ij} - \frac{1}{S^2(t)} \chi_{ik} \chi^k{}_j \quad (2.38)$$

which by (2.16), (2.24), (2.26) and (2.27) gives

$$\begin{aligned} E_{ij} = & -\frac{1}{4} S(t) \{ \partial_t (S(t) \partial_t \delta h_{ij}) - \frac{1}{3} h_{ij} h^{km} \partial_t (S(t) \partial_t \delta h_{km}) \} \\ & + \frac{1}{2} (\delta^{(3)} R_{ij} - 2k \delta h_{ij} - \frac{1}{3} h_{ij} \delta^{(3)} R) \\ & + \frac{1}{8} S^2(t) \{ h^{km} (\partial_t \delta h_{km}) (\partial_t \delta h_{ij}) - \frac{1}{3} h^{np} h^{km} (\partial_t \delta h_{km}) (\partial_t \delta h_{np}) h_{ij} \} . \end{aligned} \quad (2.39)$$

To compute B_{ab} one may proceed directly from (2.35) to obtain

$$B_{ab} = -u^e \eta_e{}^{cd}{}_{a} (\chi_{bd;c} - \frac{1}{2} g_{cb} R_{fd} u^f) \quad (2.40)$$

which by the symmetry of B_{ab} gives

$$B_{ab} = -u^e \eta_e{}^{cd}{}_{(a} \chi_{b)d;c} . \quad (2.41)$$

By means of (2.16) one thus has

$$B_{ij} = -\frac{1}{4} S(t) \{ {}^{(3)} \eta^{lm}{}_i (\partial_t \delta h_{jm})_{||l} + {}^{(3)} \eta^{lm}{}_j (\partial_t \delta h_{im})_{||l} \} \quad (2.42)$$

where

$${}^{(3)}\eta_{ijk} = \frac{1}{S^3(t)} u^a \eta_{aijk} \quad (2.43)$$

is the alternating tensor for h_{ij} .

The calculations so far have all been exact. However, in order to facilitate further progress, the perturbation $\delta \mathbf{h}$ will be treated as a power series in κ that vanishes to zeroth order, and the perturbed metric \mathbf{g} will be studied only to first order in κ . Note that the spatial curvature constant k will not be treated as a power series in κ so, for example, terms with the coefficient $k\kappa$ will not be disregarded in first order approximations. This contrasts, in particular, with the work of Holz & Wald (1998).

3. REDSHIFT AND EMISSION TIME

Suppose a comoving observer with world line $\{\mathbf{x} = \mathbf{x}_1\}$ in the perturbed space-time makes an observation at time $t = t_1$ of a comoving source with redshift z_1 . The image seen by the observer is formed by a congruence of null geodesics centred on a null geodesic γ connecting a point (t_0, \mathbf{x}_0) on the source world line $\{\mathbf{x} = \mathbf{x}_0\}$ to the observation point (t_1, \mathbf{x}_1) on the observer's world line $\{\mathbf{x} = \mathbf{x}_1\}$. For comparison, consider an observer in the unperturbed space-time who, at the same observation point (t_1, \mathbf{x}_1) , makes an observation of a comoving source with the same redshift z_1 . In this case let $\overset{\circ}{\gamma}$ be the null geodesic, $\{\mathbf{x} = \overset{\circ}{\mathbf{x}}_0\}$ the source world line and $\overset{\circ}{t}_0$ the time of emission. For definiteness let $\overset{\circ}{\gamma}$ be chosen such that its spatial direction at (t_1, \mathbf{x}_1) coincides with that of γ .

The immediate problem is to compute the perturbation $\delta t_0 = t_0 - \overset{\circ}{t}_0$ in the time of emission. For $\overset{\circ}{\mathbf{g}}$ the emission time $\overset{\circ}{t}_0$ is determined implicitly by

$$1 + z_1 = \frac{S(t_1)}{S(\overset{\circ}{t}_0)} \quad (3.1)$$

which for nearby sources gives

$$t_1 - \overset{\circ}{t}_0 = \frac{z_1}{\overset{\circ}{H}(t_1)} + O(z_1^2) \quad \text{for } z_1 \ll 1. \quad (3.2)$$

For \mathbf{g} one may determine t_0 as follows. Since γ is a null geodesic one has

$$S^2(t) h_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 1 \quad (3.3)$$

$$\frac{d^2 x^a}{d\lambda^2} + \Gamma_{bc}^a \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda} = 0 \quad (3.4)$$

where λ is an affine parameter along γ with value λ_0 at (t_0, \mathbf{x}_0) and value λ_1 at (t_1, \mathbf{x}_1) . By means of (3.3) the t component of (3.4) gives

$$\frac{d}{dt} \ln \left(\frac{dt}{d\lambda} \right) + \frac{\dot{S}(t)}{S(t)} + \kappa \nu_\gamma(t) = 0 \quad (3.5)$$

where

$$\kappa \nu_\gamma(t) := \frac{1}{2} S^2(t) (\partial_t \delta h_{ij}) \frac{dx^i}{dt} \frac{dx^j}{dt} \quad (3.6)$$

is defined along γ . The redshift as observed at time t is given by the standard formula

$$1 + z(t) = \frac{\left(\frac{dt}{d\lambda}\right|_{t_0}}{\left(\frac{dt}{d\lambda}\right|_t)} \quad (3.7)$$

which combines with (3.5) to give

$$\frac{d}{dt} \ln(1 + z(t)) = \frac{\dot{S}(t)}{S(t)} + \kappa \nu_\gamma(t) . \quad (3.8)$$

Integration of (3.8) from t_0 to t_1 yields

$$1 + z_1 = \frac{S(t_1)}{S(t_0)} \exp\left(\int_{t=t_0}^{t=t_1} \kappa \nu_\gamma(t) dt\right) \quad (3.9)$$

which may be regarded as an implicit equation for t_0 in terms of z_1 and t_1 . To first order in κ equation (3.9) gives

$$\delta t_0 = \frac{1}{H(t_0)} \int_{t=\overset{\circ}{t}_0}^{t=t_1} \kappa \nu_\gamma(t) dt + O(\kappa^2) \quad (3.10)$$

by means of (3.1), with $\overset{\circ}{t}_0$ is given implicitly in terms of z_1 and t_1 by (3.1), and with $\kappa \nu_\gamma(t)$ defined by an equation corresponding to (3.6) but for $\overset{\circ}{\gamma}$ in place of γ : clearly one has $\kappa \nu_\gamma(t) = \kappa \nu_\gamma(t) + O(\kappa^2)$. For nearby sources (3.10) gives

$$\delta t_0 = \frac{\kappa \nu_\gamma(t_1)}{\overset{\circ}{H}^2(t_1)} z_1 + O(z_1^2) + O(\kappa^2) \quad \text{for } z_1 \ll 1 \quad (3.11)$$

by means of (3.2).

One may define a spatial proper distance of the source from the observer, with respect to the metric \mathbf{g} , by

$$s_0 := c \int_{\lambda=\lambda_0}^{\lambda=\lambda_1} (S^2(t) h_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda})^{1/2} d\lambda = c(t_1 - t_0) \quad (3.12)$$

where (3.3) has been employed on the right. For nearby sources (3.2) and (3.11) combine to give

$$t_1 - t_0 = \frac{z_1}{\overset{\circ}{H}(t_1)} \left(1 - \frac{\kappa \nu_\gamma(t_1)}{\overset{\circ}{H}(t_1)}\right) + O(z_1^2) + O(\kappa^2) \quad \text{for } z_1 \ll 1 \quad (3.13)$$

so, for such sources, the redshift z_1 of the source is expressible as a function of its spatial proper distance s_0 according to

$$z_1(s_0) = \frac{\overset{\circ}{H}(t_1) s_0}{c} \left(1 + \frac{\kappa \nu_\gamma(t_1)}{\overset{\circ}{H}(t_1)}\right) + O(s_0^2) + O(\kappa^2) \quad \text{for } z_1 \ll 1 . \quad (3.14)$$

Notice that in the case $\nu_\gamma(t_1) = O(\kappa)$ equation (3.14) agrees, to first order in z_1 and κ , with the corresponding formula for the unperturbed FRW space-time.

4. LUMINOSITY DISTANCE AND APPARENT MAGNITUDE

Suppose now that, in the perturbed space-time, the source with world line $\{\mathbf{x} = \mathbf{x}_0\}$ radiates uniformly in all directions with power P .

Let $k^a := \frac{dx^a}{d\lambda}$ be the tangent to γ and let \mathbf{m} be a complex vector at (t_0, \mathbf{x}_0) satisfying $\mathbf{m} \cdot \bar{\mathbf{m}} = 1$, $\mathbf{m} \cdot \mathbf{k} = \mathbf{m} \cdot \bar{\mathbf{m}} = \mathbf{m} \cdot \mathbf{u} = 0$. In order to maintain these conditions along γ one requires that \mathbf{m} be propagated along γ according to

$$\nabla_{\mathbf{k}} \mathbf{m} = - \left(\frac{\mathbf{m} \cdot \nabla_{\mathbf{k}} \mathbf{u}}{\mathbf{u} \cdot \mathbf{k}} \right) \mathbf{k} . \quad (4.1)$$

The congruence of all the null geodesics emanating from (t_0, \mathbf{x}_0) gives rise to a family of Jacobi fields along γ with expansion $\varrho := k_{a;b} \bar{m}^a m^b$ and shear $\varsigma := k_{a;b} \bar{m}^a \bar{m}^b$ with respect to the 2-frame $\{\mathbf{m}, \bar{\mathbf{m}}\}$ and the affine parameter λ . Since the congruence is irrotational, the imaginary part of ϱ is zero. The standard propagation equations for ϱ and ς therefore reduce to

$$\frac{d}{d\lambda} \varrho = -\varrho^2 - \varsigma \bar{\varsigma} - \Theta \quad (4.2)$$

$$\frac{d}{d\lambda} \varsigma = -2\varsigma \varrho - \Psi \quad (4.3)$$

where

$$\Theta := \frac{1}{2} R_{ab} k^a k^b = \frac{1}{2} \kappa T_{ab} k^a k^b \quad (4.4)$$

$$\Psi := R_{abcd} k^a k^c \bar{m}^b \bar{m}^d = C_{abcd} k^a k^c \bar{m}^b \bar{m}^d \quad (4.5)$$

are the Ricci and Weyl scalars respectively. (The fact that \mathbf{m} is propagated along γ according to (4.1), rather than being parallelly propagated as is more usual, has no effect on the equations (4.2) and (4.3).)

Consider a narrow beam of light rays, centred upon γ , from the source point (t_0, \mathbf{x}_0) . Let ΔA and I be the cross-sectional area and apparent luminosity of the beam as determined by comoving observers situated along γ . A standard argument from photon conservation gives that the quantity $(1+z)^2 I \Delta A$ is constant along γ . By means of $\varrho = \frac{1}{2} \frac{d}{d\lambda} \ln \Delta A$ one thus obtains

$$\varrho = \frac{d}{d\lambda} \ln(I^{-1/2} \frac{dt}{d\lambda}) = \frac{d}{d\lambda} \ln(I^{-1/2} (1+z)^{-1}) . \quad (4.6)$$

In terms of the quantities

$$J(\lambda) := I^{-1/2} (1+z)^{-1} \quad (4.7)$$

$$\xi(\lambda) := J^2(\lambda) \left(\frac{dt}{d\lambda} \right)^{-1} \varsigma(\lambda) \quad (4.8)$$

equations (4.2) and (4.3) become

$$\frac{d^2}{d\lambda^2} J(\lambda) + J^{-3}(\lambda) \xi(\lambda) \bar{\xi}(\lambda) \left(\frac{dt}{d\lambda} \right)^2 + J(\lambda) \Theta = 0 \quad (4.9)$$

$$\frac{d}{d\lambda} \left(\frac{dt}{d\lambda} \xi(\lambda) \right) + J^2(\lambda) \Psi = 0 . \quad (4.10)$$

Substituting for λ in terms of t in (4.9) and (4.10) by means of (3.5) one thus obtains

$$\frac{d^2}{dt^2}J(t) - \left(\frac{\dot{S}(t)}{S(t)} + \kappa\nu_\gamma(t) \right) \frac{d}{dt}J(t) + \xi(t)\bar{\xi}(t)J^{-3}(t) + \kappa\tau_\gamma(t)J(t) = 0 \quad (4.11)$$

$$\frac{d}{dt}\xi(t) - \left(\frac{\dot{S}(t)}{S(t)} + \kappa\nu_\gamma(t) \right) \xi(t) + J^2(t)\kappa\psi_\gamma(t) = 0 \quad (4.12)$$

for

$$\kappa\tau_\gamma(t) := \frac{1}{2}\kappa T_{ab} \frac{dx^a}{dt} \frac{dx^b}{dt} \quad (4.13)$$

$$\kappa\psi_\gamma(t) := C_{abcd} \frac{dx^a}{dt} \frac{dx^c}{dt} \bar{m}^b \bar{m}^d \quad (4.14)$$

$$= 2E_{ij}\bar{m}^i\bar{m}^j + \frac{2}{S^2(t)} {}^{(3)}\eta^k{}_{ij} B_{lk}\bar{m}^j\bar{m}^l \frac{dx^i}{dt} . \quad (4.15)$$

The initial conditions for (4.11) and (4.12) are

$$\left. \begin{aligned} \xi(\overset{\circ}{t}_0) &= 0 \\ J(\overset{\circ}{t}_0) &= 0 \\ \frac{d}{dt} \Big|_{t=\overset{\circ}{t}_0} J(t) &= c\sqrt{\frac{4\pi}{P}} \end{aligned} \right\} . \quad (4.16)$$

To first order in κ , the solutions to (4.11) and (4.12), subject to the initial conditions (4.16), are

$$J(t) = \sqrt{\frac{4\pi}{P}} \frac{c}{S(t_0)} \int_{t'=t_0}^{t'=t} S(t')(1 + \kappa\mu_\gamma^\circ(t')) dt' + O(\kappa^2) \quad (4.17)$$

$$\xi(t) = -\frac{4\pi c^2}{P} \frac{S(t)}{S(t_0)} \int_{t'=t_0}^{t'=t} \frac{\Sigma^2(t_0, t')}{S(t')} \kappa\psi_\gamma^\circ(t') dt' + O(\kappa^2) \quad (4.18)$$

for

$$\Sigma(t_0, t) := \int_{t'=t_0}^{t'=t} S(t') dt' \quad (4.19)$$

and

$$\kappa\mu_\gamma^\circ(t) := \int_{t'=t_0}^{t'=t} (\kappa\nu_\gamma^\circ(t') - \frac{\Sigma(t_0, t')}{S(t')} \kappa\tau_\gamma^\circ(t')) dt' \quad (4.20)$$

where $\kappa\tau_\gamma^\circ(t)$ and $\kappa\psi_\gamma^\circ(t)$ are defined by equations analogous to (4.13) and (4.14) for $\overset{\circ}{\gamma}$ in place of γ . Clearly one has $\kappa\tau_\gamma^\circ(t) = \kappa\tau_\gamma(t) + O(\kappa^2)$ and $\kappa\psi_\gamma^\circ(t) = \kappa\psi_\gamma(t) + O(\kappa^2)$.

From (4.8), (4.17) and (4.18) one obtains

$$\varsigma(t) = -\frac{S(\overset{\circ}{t}_0)S(t)}{\Sigma^2(\overset{\circ}{t}_0, t)} \frac{dt}{d\lambda} \int_{t'=\overset{\circ}{t}_0}^{t'=t} \frac{\Sigma^2(\overset{\circ}{t}_0, t')}{S(t')} \kappa\psi_\gamma^\circ(t') dt' + O(\kappa^2) \quad (4.21)$$

for the shear as a function of t .

Writing $t_0 = \overset{\circ}{t}_0 + \delta t_0$ and setting $t = t_1$ in (4.17) one obtains

$$I^{-1/2}(z_1) = c\sqrt{\frac{4\pi}{P}} \frac{(1+z_1)^2}{S(t_1)} \left\{ \Sigma(\overset{\circ}{t}_0, t_1) \left(\frac{S^2(\overset{\circ}{t}_0)}{\dot{S}(\overset{\circ}{t}_0)} \Sigma(\overset{\circ}{t}_0, t_1) \right) \int_{t=\overset{\circ}{t}_0}^{t=t_1} \kappa \nu_\gamma^\circ(t) dt \right. \\ \left. + \int_{t=\overset{\circ}{t}_0}^{t=t_1} S(t) \kappa \mu_\gamma^\circ(t) dt \right\} + O(\kappa^2) \quad (4.22)$$

by means of (3.10) and 4.19), where $\overset{\circ}{t}_0$ is given in terms of t_1 and z_1 by (3.1). From this one obtains

$$I^{-1/2}(z_1) = c\sqrt{\frac{4\pi}{P}} \frac{(1+z_1)^2}{S(t_1)} \left\{ \Sigma(\overset{\circ}{t}_0, t_1) - \frac{S^2(\overset{\circ}{t}_0)}{\dot{S}(t_1)} \int_{t=\overset{\circ}{t}_0}^{t=t_1} \kappa \nu_\gamma^\circ(t) dt \right. \\ \left. - \int_{t=\overset{\circ}{t}_0}^{t=t_1} \Sigma(\overset{\circ}{t}_0, t) (\kappa \nu_\gamma^\circ(t) - \frac{\Sigma(\overset{\circ}{t}_0, t)}{S(t)} \kappa \tau_\gamma^\circ(t)) dt \right. \\ \left. - \Sigma(\overset{\circ}{t}_0, t_1) \int_{t=\overset{\circ}{t}_0}^{t=t_1} \frac{\Sigma(\overset{\circ}{t}_0, t)}{S(t)} \kappa \tau_\gamma^\circ(t) dt \right\} + O(\kappa^2) \quad (4.23)$$

by means of (4.20), (4.19) and an integration by parts. For nearby sources (4.23) gives

$$I^{-1/2}(z_1) = \sqrt{\frac{4\pi}{P}} \frac{cz_1}{\overset{\circ}{H}(t_1)} \left(1 - \frac{\kappa \nu_\gamma^\circ(t_1)}{\overset{\circ}{H}(t_1)} \right) + O(z_1^2) + O(\kappa^2) \quad \text{for } z_1 \ll 1 \quad (4.24)$$

by means of (3.13). The equation corresponding to (4.23) for the unperturbed metric $\overset{\circ}{\mathbf{g}}$ is

$$\overset{\circ}{I}^{-1/2}(z_1) = c\sqrt{\frac{4\pi}{P}} \frac{(1+z_1)^2}{S(t_1)} \Sigma(\overset{\circ}{t}_0, t_1) . \quad (4.25)$$

From (4.21), the total shear of the image at the observation point (t_1, \mathbf{x}_1) is given by

$$\int_{\lambda=\lambda_0}^{\lambda=\lambda_1} \varsigma d\lambda = - \int_{t=\overset{\circ}{t}_0}^{t=t_1} \frac{S(\overset{\circ}{t}_0)S(t)}{\Sigma^2(\overset{\circ}{t}_0, t)} \left(\int_{t'=\overset{\circ}{t}_0}^{t'=t} \frac{\Sigma^2(\overset{\circ}{t}_0, t')}{S(t')} \kappa \psi_\gamma^\circ(t') dt' \right) dt + O(\kappa^2) . \quad (4.26)$$

The luminosity distance of the source is defined by

$$d_L := \sqrt{\frac{P}{4\pi I(z_1)}} \quad (4.27)$$

which, for nearby sources, combines with (4.24) to give

$$d_L = \frac{cz_1}{\overset{\circ}{H}(t_1)} \left(1 - \frac{\kappa \nu_\gamma^\circ(t_1)}{\overset{\circ}{H}(t_1)} \right) + O(z_1^2) + O(\kappa^2) \quad \text{for } z_1 \ll 1 . \quad (4.28)$$

Note that in the case $\nu_\gamma(t_1) = O(\kappa)$ this agrees, to first order in z_1 and κ , with the corresponding formula for the unperturbed FRW space-time. From (4.28) and

(3.14) one has

$$d_L = s_0 + O(s_0^2) + O(\kappa^2) \quad (4.29)$$

which gives the acceptable result that the spatial proper distance and luminosity distance agree to first order in s_0 and κ , independent of the value of $\nu_\gamma^\circ(t_1)$. This holds for both the perturbed metric \mathbf{g} and the unperturbed metric $\mathring{\mathbf{g}}$.

The apparent magnitude of the source at $\mathbf{x} = \mathbf{x}_0$ relative to a reference source at $\mathbf{x} = \mathbf{x}_{ref}$ is defined by

$$m := m_{ref} + \frac{5}{2} \log_{10} \frac{I_{ref}}{I} \quad (4.30)$$

where I and I_{ref} are the apparent luminosities of the respective sources. The reference source is taken to have power P_{ref} and to be at a spatial proper distance $s_{0,ref}$ that is small on a cosmological scale (10 parsecs is conventional). For the unperturbed metric $\mathring{\mathbf{g}}$, a source of apparent luminosity \mathring{I} has an apparent magnitude

$$\mathring{m} = \mathring{m}_{ref} + \frac{5}{2} \log_{10} \frac{\mathring{I}_{ref}}{\mathring{I}} \quad (4.31)$$

relative to a source of apparent luminosity \mathring{I}_{ref} . The sources will again be taken to have powers P and P_{ref} respectively, with the reference source at the same spatial proper distance $s_{0,ref}$ as for the perturbed metric \mathbf{g} . Let $m_{ref} = \mathring{m}_{ref}$. The objective sources for \mathbf{g} and $\mathring{\mathbf{g}}$ are both taken to be at redshift z_1 . From (4.30) and (4.31) one obtains

$$\delta m(z_1) = \frac{5}{2 \ln 10} \left\{ \lim_{s \rightarrow 0} \ln \frac{I_{ref}(s)}{\mathring{I}_{ref}(s)} - \ln \frac{I(z_1)}{\mathring{I}(z_1)} \right\} \quad (4.32)$$

in the limit $s_{0,ref} \rightarrow 0$. By (4.29), the first term in the braces is just $\ln(P/P_{ref})$ to first order in κ . By (4.23) and (4.25) one thus obtains

$$\begin{aligned} \delta m(z_1) = & \frac{5}{\ln 10} \left\{ -\frac{S^2(\mathring{t}_0)}{\mathring{S}(t_1)\mathring{\Sigma}(\mathring{t}_0, t_1)} \int_{t=\mathring{t}_0}^{t=t_1} \kappa \nu_\gamma^\circ(t) dt - \int_{t=\mathring{t}_0}^{t=t_1} \frac{\mathring{\Sigma}(\mathring{t}_0, t)}{\mathring{S}(t)} \kappa \delta \tau_\gamma^\circ(t) dt \right. \\ & \left. - \frac{1}{\mathring{\Sigma}(\mathring{t}_0, t_1)} \int_{t=\mathring{t}_0}^{t=t_1} \mathring{\Sigma}(\mathring{t}_0, t) (\kappa \nu_\gamma^\circ(t) - \frac{\mathring{\Sigma}(\mathring{t}_0, t)}{\mathring{S}(t)} \kappa \delta \tau_\gamma^\circ(t)) dt \right\} + O(\kappa^2) \end{aligned} \quad (4.33)$$

for the correction to the apparent magnitude-redshift relation. Note that this correction depends on the perturbation $\delta \mathbf{h}$ only through the functions $\kappa \nu_\gamma^\circ(t)$ and $\kappa \delta \tau_\gamma^\circ(t)$.

For nearby sources (4.33) gives

$$\delta m(z_1) = -\frac{5}{\ln 10} \frac{\kappa \nu_\gamma^\circ(t_1)}{\mathring{H}(t_1)} + O(z_1) + O(\kappa^2) \quad \text{for } z \ll 1 \quad (4.34)$$

for the apparent magnitude-redshift correction. This evidently vanishes at $z_1 = 0$ to first order in κ in the case $\nu_\gamma(t_1) = O(\kappa)$.

It is evident from the analysis in the present and preceding sections that the equation

$$\kappa \nu_\gamma(t_1) = 0 + O(\kappa^2) \quad (4.35)$$

may be interpreted as a condition on the perturbation which, as regards local optical properties, ensures that the background FRW metric $\overset{\circ}{\mathbf{g}}$ is a best fit to the perturbed metric \mathbf{g} , in the direction of $\dot{\gamma}(t_1)$, at the observation point (t_1, \mathbf{x}_1) . A requirement that (4.35) holds for all null geodesics γ through (t_1, \mathbf{x}_1) is, by (3.6), equivalent to the condition

$$\partial_t \delta h_{ij}(t_1, \mathbf{x}_1) = 0 + O(\kappa^2) . \quad (4.36)$$

In general this represent a physical constraint on the perturbation at (t_1, \mathbf{x}_1) since there is, in general, no freedom to choose a new time slicing such that the 3-metric \mathbf{h} is intrinsic to the level surfaces of the new time. However, for the specific class of perturbations to be introduced in §5, there is just such a freedom, at least to first order in κ , which may be exploited to ensure that (4.36) does hold.

5. NEWTONIAN PERTURBATIONS

In order to describe the matter distribution of the cosmos, physical considerations suggest that one seeks a perturbed 3-metric \mathbf{h} such that the corresponding space-time metric \mathbf{g} has an energy tensor of the form

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab} \quad (5.1)$$

describing a perfect fluid with velocity $u^a := t^a$, density ρ and pressure p . It is conventional to define a dimensionless density parameter by

$$\Omega := \frac{\kappa\rho}{3H^2} . \quad (5.2)$$

Note that all of ρ , H and Ω depend on all four coordinates. With regard to the pressure, since the form of (2.13) implies that u^a is geodesic, the conservation equation for (5.1) implies $h_a{}^b p_{;b} = 0$ and hence $h_a{}^b (\delta p)_{;b} = 0$ under the assumption that $\rho + p$ is nowhere zero. This leads one to consider the particular case $\delta p = 0$ which describes perturbations arising from the addition or removal of comoving dust.

The full non-linear problem presents formidable difficulties, although one does have by (5.1) and (2.25) that δh_{ij} satisfies the simple equation

$$h^{jk}(\partial_t \delta h_{k[i} \delta h_{j]}) = 0 . \quad (5.3)$$

In order to make progress, only a linear approximation to a solution will be sought. Specifically the problem is to obtain δh_{ij} , regarded as a power series in κ , vanishing to zeroth order, such that the energy tensor of g_{ab} has the form

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab} + O(\kappa) . \quad (5.4)$$

Consider a perturbation of the 3-metric $\overset{\circ}{h}_{ij}$ of the form

$$\delta h_{ij} = F(t)\kappa\Phi_{||ij} + G(t)\kappa\Phi h_{ij} + O(\kappa^2) \quad (5.5)$$

for functions $F(t)$, $G(t)$ and a t -independent scalar field $\Phi(\mathbf{x})$. The form of (5.5) corresponds to the synchronous gauge perturbations considered by Mukhanov et al. (1992, p.216). It will be convenient, although not necessary, to regard $F(t)$ and $G(t)$ as power series in κ even though only the zeroth order terms will be of significance. Note that although δh_{ij} enters into the right side of (5.5) through the term $\Phi_{||ij}$, this is not significant in the linear approximation since $\Phi_{||ij}$ and $\overset{\circ}{\Phi}_{||ij}$ agree to zeroth order in κ .

From first principles one has

$$\delta^{(3)} R_{ij} - 2k\delta h_{ik} = -\frac{1}{2}G(t)\{\kappa\Phi_{||ij} + (\kappa^{(3)}\Delta\Phi + 4k\kappa\Phi)h_{ij}\} + O(\kappa^2) \quad (5.6)$$

which by (2.28) gives

$$\delta^{(3)} R = -2G(t)(\kappa^{(3)}\Delta\Phi + 3k\kappa\Phi) + O(\kappa^2) . \quad (5.7)$$

Substitution of (5.5) into (2.30), (2.31) and (2.32) yields

$$\kappa T_{tt} = \kappa \overset{\circ}{\rho} + \left(\frac{\dot{S}(t)\dot{F}(t)}{S^2(t)} - \frac{G(t)}{S^2(t)}\right)\kappa^{(3)}\Delta\Phi + 3\left(\frac{\dot{S}(t)\dot{G}(t)}{S(t)} - \frac{kG(t)}{S^2(t)}\right)\kappa\Phi + O(\kappa^2) \quad (5.8)$$

$$\kappa T_{it} = \kappa T_{ti} = (k\dot{F}(t) - \dot{G}(t))\kappa\Phi_{||i} + O(\kappa^2) \quad (5.9)$$

$$\begin{aligned} \kappa T_{ij} = & \{S^2(t)\kappa \overset{\circ}{p} + \frac{1}{2}(G(t) - \frac{1}{S(t)}(S^3(t)\dot{F}(t))')\}\kappa^{(3)}\Delta\Phi \\ & + (kG(t) - \frac{1}{S(t)}(S^3(t)\dot{G}(t))')\kappa\Phi\}h_{ij} \\ & + \frac{1}{2}\left(\frac{1}{S(t)}(S^3(t)\dot{F}(t))' - G(t)\right)\kappa\Phi_{||ij} + O(\kappa^2) \end{aligned} \quad (5.10)$$

by means of (5.7), (2.11) and (5.6).

For κT_{it} and the trace-free part of κT_{ij} to vanish to first order in κ , in accordance with (5.4), it suffices to require that $F(t)$ and $G(t)$ satisfy

$$G(t) = \frac{1}{S(t)}(S^3(t)\dot{F}(t))' + O(\kappa) \quad (5.11)$$

$$\dot{G}(t) = k\dot{F}(t) + O(\kappa) . \quad (5.12)$$

From these one obtains

$$(S(t)(S^2(t)\dot{F}(t))')' = 0 + O(\kappa) \quad (5.13)$$

by means of (2.9), and hence

$$S(t)(S^2(t)\dot{F}(t))' = C + O(\kappa) \quad (5.14)$$

for some constant C .

If C were chosen to vanish to zeroth order in κ then, by (5.14) and (5.11), both $(S^2(t)\dot{F}(t))'$ and $G(t) - \dot{S}(t)S(t)\dot{F}(t)$ would vanish to zeroth order in κ and so, as will be evident from (6.8) in §6, **g** would be an FRW metric to first order in κ irrespective of the function $\Phi(\mathbf{x})$. To avoid this uninteresting case C will be chosen to be non-zero to zeroth order in κ . One may then normalise $F(t)$ to give

$$C = 1 . \quad (5.15)$$

Equations (5.11), (5.14) and (5.15) give

$$(S^2(t)\dot{F}(t))' = \frac{1}{S(t)} + O(\kappa) \quad (5.16)$$

$$G(t) - \dot{S}(t)S(t)\dot{F}(t) = \frac{1}{S(t)} + O(\kappa) . \quad (5.17)$$

These are equivalent to equations (5.11), (5.12) and (5.15) by virtue of (2.9).

By means of (5.16) and (5.17), equations (5.8), (5.9) and (5.10) combine to give that T^{ab} has the required form (5.4) for

$$\delta\rho = -\frac{1}{S^3(t)}({}^{(3)}\Delta\Phi + 3k\Phi) + O(\kappa) \quad (5.18)$$

$$\delta p = 0 + O(\kappa) . \quad (5.19)$$

From (5.18) one has

$$\partial_t(S^3(t)\delta\rho) = 0 + O(\kappa) \quad (5.20)$$

as one would expect since the perturbation has a dust equation of state.

From (5.5), (5.12) and (5.18) one has

$$h^{ij}\partial_t\delta h_{ij} = -S^3(t)\dot{F}(t)\kappa\delta\rho + O(\kappa^2) \quad (5.21)$$

whereby (2.21), (2.22) and (2.23) give

$$\delta H = -\frac{1}{6}S^3(t)\dot{F}(t)\kappa\delta\rho + O(\kappa^2) \quad (5.22)$$

$$\delta q = \frac{1}{2}\{S(t)(S^2(t)\dot{F}(t))' + 2S^3(t)\dot{S}(t)\dot{F}(t)(1 + \overset{\circ}{q}(t))\}\delta\Omega + O(\kappa^2) \quad (5.23)$$

$$\delta {}^{(3)}R = 2S^3(t)G(t)\kappa\delta\rho + O(\kappa^2) \quad (5.24)$$

with the help of (5.2) and (5.17).

It is straightforward to check, by means of (2.42) and (2.11), that the perturbed space-time metric \mathbf{g} defined by (2.13), (2.14) and (5.5) is silent to first order in κ in the sense of

$$B_{ij} = 0 + O(\kappa^2) . \quad (5.25)$$

The electric part of the Weyl tensor of \mathbf{g} is, by (2.39), (5.6), (5.7), (5.16) and (5.17), given by

$$E_{ij} = -\frac{1}{2S(t)}(\kappa\Phi_{\parallel ij} - \frac{1}{3}h_{ij}\kappa {}^{(3)}\Delta\Phi) + O(\kappa^2) . \quad (5.26)$$

By (5.4) and the analogue of (4.13) for $\overset{\circ}{\gamma}$, the function $\delta\tau_{\overset{\circ}{\gamma}}(t)$ in (4.33) is given by

$$\delta\tau_{\overset{\circ}{\gamma}}(t) = \frac{1}{2}\delta\rho(\overset{\circ}{\gamma}(t)) + O(\kappa) . \quad (5.27)$$

An equation for the function $\nu_{\overset{\circ}{\gamma}}^s(t)$ in (4.33) will be given in §6. By (5.25), (5.26) and the analogue of (4.15) for $\overset{\circ}{\gamma}$ the function $\psi_{\overset{\circ}{\gamma}}(t)$ in equation (4.26) is given by

$$\psi_{\overset{\circ}{\gamma}}(t) = -\frac{1}{S(t)}\Phi_{\parallel ij}^{\overset{\circ}{\gamma}}\bar{m}^i\bar{m}^j + O(\kappa) \quad (5.28)$$

where \bar{m}^i is the analogue of m^i for $\overset{\circ}{\gamma}$.

Space-times satisfying the condition $B_{ab} = 0$ have been termed ‘silent’ by some authors and ‘Newtonian-like’ by others. However it is known that the vanishing of B_{ab} can be conserved in time only in specialised cases (Maartens *et al.* 1998). In any second or higher order study of perturbations of FRW cosmologies one would therefore not expect B_{ab} to vanish to any higher than first order in κ .

6. THE GAUGE CONDITION

The functions $F(t)$ and $G(t)$, which enter explicitly into the perturbed metric \mathbf{g} , are not determined uniquely by (5.16) and (5.17) since these equations allow two freely specifiable constants of integration. In alternative terminology, there is a gauge freedom

$$F(t) \rightarrow F(t) + A + A_0 \int_{t'=t}^{t'=t_1} \frac{dt'}{S^2(t)} \quad (6.1)$$

$$G(t) \rightarrow G(t) - A_0 \frac{\dot{S}(t)}{S(t)} \quad (6.2)$$

where A_0 and A are real constants. In order to understand the meaning of this freedom, consider the metric \mathbf{g} , as given by (2.13), (2.14) and (5.5), expressed with respect to the coordinate system employed in (2.2):

$$\mathbf{g} = -dt^2 + S^2(t) \left(1 + \frac{kr^2}{4}\right)^{-2} (1 + G(t)\kappa\Phi(\mathbf{x})) \sum_{i=1}^3 (dx^i)^2 + S^2(t)F(t)\kappa\Phi(\mathbf{x})_{\parallel ij} dx^i dx^j + O(\kappa^2) . \quad (6.3)$$

In terms of new coordinates $(\tilde{t}, \tilde{\mathbf{x}})$ defined by

$$\tilde{t} := t + \frac{1}{2}F^*(t)\kappa\Phi(\mathbf{x}) \quad (6.4)$$

$$\tilde{\mathbf{x}}^i := x^i + \frac{1}{2}F(t) \left(1 + \frac{kr^2}{4}\right)^2 \kappa\partial_i\Phi(\mathbf{x}) , \quad (6.5)$$

for an as yet unspecified function $F^*(t)$, the metric \mathbf{g} assumes the form

$$\mathbf{g} = -(1 - \dot{F}^*(\tilde{t})\kappa\Phi(\tilde{\mathbf{x}}))d\tilde{t}^2 + (F^*(\tilde{t}) - S^2(\tilde{t})\dot{F}(\tilde{t}))\kappa\partial_{\tilde{\mathbf{x}}^i}\Phi(\tilde{\mathbf{x}})d\tilde{x}^i d\tilde{t} + \frac{S^2(\tilde{t})}{(1 + \frac{k\tilde{r}^2}{4})^2} (1 + (G(\tilde{t}) - \frac{\dot{S}(\tilde{t})}{S(\tilde{t})}F^*(\tilde{t}))\kappa\Phi(\tilde{\mathbf{x}})) \sum_{i=1}^3 (d\tilde{x}^i)^2 + O(\kappa^2) . \quad (6.6)$$

With the choice

$$F^*(t) = S^2(t)\dot{F}(t) \quad (6.7)$$

the metric \mathbf{g} assumes the diagonal form

$$\mathbf{g} = -(1 - (S^2(\tilde{t})\dot{F}(\tilde{t}))\kappa\Phi(\tilde{\mathbf{x}}))d\tilde{t}^2 + \frac{S^2(\tilde{t})}{(1 + \frac{k\tilde{r}^2}{4})^2} (1 + (G(\tilde{t}) - \dot{S}(\tilde{t})S(\tilde{t})\dot{F}(\tilde{t}))\kappa\Phi(\tilde{\mathbf{x}})) \sum_{i=1}^3 (d\tilde{x}^i)^2 + O(\kappa^2) . \quad (6.8)$$

No assumptions about the t -dependence of $F(t)$ and $G(t)$ have yet been employed. It is evident that if $(S^2(t)\dot{F}(t))$ and $G(t) - \dot{S}(t)S(t)\dot{F}(t)$ were both to vanish to zeroth order in κ then, as quoted in §5, \mathbf{g} would be an FRW metric to first order in κ .

For $F(t)$ and $G(t)$ satisfying (5.16) and (5.17), equation (6.8) reduces to

$$\mathbf{g} = -\left(1 - \frac{\kappa\Phi(\tilde{\mathbf{x}})}{S(\tilde{t})}\right)d\tilde{t}^2 + \frac{S^2(\tilde{t})}{(1 + \frac{k\tilde{r}^2}{4})^2} \left(1 + \frac{\kappa\Phi(\tilde{\mathbf{x}})}{S(\tilde{t})}\right) \sum_{i=1}^3 (d\tilde{x}^i)^2 + O(\kappa^2) \quad (6.9)$$

which is the metric of Newman & McVittie (1982). Note that the curves $\{\tilde{\mathbf{x}} = \text{const.}\}$ have unit tangent

$$\tilde{u}_a = - \left(1 + \frac{\kappa \Phi(\tilde{\mathbf{x}})}{2S(\tilde{t})} \right) \tilde{t}_{;a} + O(\kappa^2) \quad (6.10)$$

and so are non-geodesic to first order in κ . They are therefore not the world lines of freely falling observers.

The absence of $F(t)$ and $G(t)$ in (6.9) shows that the isometry class of \mathbf{g} is unaffected by the choice of integration constants for (5.16) and (5.17). Indeed the gauge transformation (6.1), (6.2) is induced by the coordinate transformation

$$t \rightarrow t + \frac{A_0}{2} \kappa \Phi(\mathbf{x}) \quad (6.11)$$

$$x^i \rightarrow x^i - \frac{1}{2} \left(A + A_0 \int_{t'=t}^{t'=t_1} \frac{dt'}{S^2(t')} \right) \left(1 + \frac{kr^2}{4} \right)^2 \kappa \partial_i \Phi(\mathbf{x}) \quad (6.12)$$

which preserves the form of the metric (6.3) for $F(t)$, $G(t)$ transforming according to (6.1) and (6.2). Moreover the transformation (6.11) of t preserves the form of (5.5) on the level surfaces of t , for a given Φ on the space-time manifold, for $F(t)$, $G(t)$ transforming according to (6.1) and (6.2). The gauge freedom (6.1), (6.2) is thus to be interpreted as a freedom to choose the time function t without violating the form of (5.5). The coordinate freedom (6.11), (6.12) corresponds to that identified by Mukhanov et al. (1992, p.216).

Although the integration constants for (5.16) and (5.17) do not affect the isometry class of \mathbf{g} to first order in κ , they nonetheless carry physical significance since they help determine the physically significant coordinate t . In order to make physically appropriate choices of these constants, consider the function $\nu_\gamma(t)$ of (3.6) along a light ray γ through the observation point (t_1, \mathbf{x}_1) . By means of (5.5), (5.12) and the null geodesic equations (3.3) and (3.4) one obtains

$$\nu_\gamma(t) = \dot{F}(t) \left\{ S(t) \frac{d}{dt} \left(S(t) \frac{d}{dt} \Phi(\gamma(t)) \right) + \Phi(\gamma(t)) \right\} + O(\kappa) . \quad (6.13)$$

By (5.5) and (5.12) the gauge condition (4.36) becomes

$$\dot{F}(t_1) = 0 . \quad (6.14)$$

This holds for a unique value of A_0 in (6.1), (6.2).

Equations (5.16), (5.17) and (6.14) give

$$\dot{F}(t) = - \frac{1}{S^2(t)} \int_{t'=t}^{t'=t_1} \frac{dt'}{S(t')} + O(\kappa) \quad (6.15)$$

$$G(t) = \frac{1}{S(t)} - \frac{\dot{S}(t)}{S(t)} \int_{t'=t}^{t'=t_1} \frac{dt'}{S(t')} + O(\kappa) . \quad (6.16)$$

Thus, to zeroth order in κ , $G(t)$ is now uniquely specified whilst $F(t)$ is determined only up to an arbitrary additive constant. By (6.11) and (6.12) this remaining gauge freedom corresponds to a coordinate transformation of the form

$$t \rightarrow t \quad (6.17)$$

$$x^i \rightarrow x^i - \frac{A}{2} \left(1 + \frac{kr^2}{4} \right)^2 \kappa \partial_i \Phi(\mathbf{x}) \quad (6.18)$$

where A is an arbitrary constant. Since this preserves the level surfaces of t it has no significance for the physical properties of the perturbed cosmology. There is therefore no need to specify the remaining integration constant of (5.16) and (5.17).

By (4.34) the gauge fixing condition (6.14) may be interpreted as a necessary and sufficient condition that there is a vanishing correction to the apparent magnitude-redshift relation, in all directions, to zeroth order in the redshift and first order in κ . By (4.28) another interpretation is that there is a vanishing correction to the luminosity distance-redshift relation, in all directions, to zeroth order in the redshift and first order in κ .

An immediate consequence of the gauge fixing condition (6.14) is, by (5.22), that the Hubble expansion H of the perturbed cosmology satisfies

$$H(t_1, \mathbf{x}) = \overset{\circ}{H}(t_1) + O(\kappa^2) \quad (6.19)$$

and so is unperturbed to first order in κ at all points of the surface $\{t = t_1\}$. By the use of (5.14) and (6.14) in (5.23) one also has

$$\delta q = \frac{1}{2} \delta \Omega + O(\kappa^2) . \quad (6.20)$$

The gauge condition (6.14) thus ensures that the perturbation (5.5) affects the deceleration parameter but not the Hubble parameter at the observation point (t_1, \mathbf{x}_1) .

Subject to the gauge fixing condition (6.14) the apparent magnitude-redshift relation is given by (4.33), with $\nu_{\gamma}^{\circ}(t)$ given by (6.13) and (6.15) and $\delta \tau_{\gamma}^{\circ}(t)$ by (5.27). Substitution for these quantities in (4.33) yields

$$\begin{aligned} \delta m(z_1) = & \frac{5}{\ln 10} \left\{ \frac{S^4(\overset{\circ}{t}_0) \dot{F}(\overset{\circ}{t}_0)}{2 \dot{S}(t_1) \Sigma(\overset{\circ}{t}_0, t_1)} \frac{d}{dt} \Big|_{t=\overset{\circ}{t}_0} \kappa \Phi(\overset{\circ}{\gamma}(t)) + \frac{S^2(\overset{\circ}{t}_0)}{\dot{S}(t_1) \Sigma(\overset{\circ}{t}_0, t_1)} \int_{t=\overset{\circ}{t}_0}^{t=t_1} \frac{\dot{S}(t)}{S^2(t)} \kappa \Phi(\overset{\circ}{\gamma}(t)) dt \right. \\ & + \frac{S^2(\overset{\circ}{t}_0)}{\dot{S}(t_1) \Sigma(\overset{\circ}{t}_0, t_1)} \left[\left(\frac{1}{S(t)} - \frac{1}{2} G(t) \right) \kappa \Phi(\overset{\circ}{\gamma}(t)) \right]_{t=\overset{\circ}{t}_0}^{t=t_1} + \frac{\kappa \Phi(\overset{\circ}{\gamma}(t_1))}{2 S(t_1)} \\ & - \frac{1}{\Sigma(\overset{\circ}{t}_0, t_1)} \int_{t=\overset{\circ}{t}_0}^{t=t_1} \left(1 - \frac{\dot{S}(t)}{S^2(t)} \Sigma(\overset{\circ}{t}_0, t) \right) \kappa \Phi(\overset{\circ}{\gamma}(t)) dt - \frac{S^3(\overset{\circ}{t}_0) \dot{F}(\overset{\circ}{t}_0)}{2 \Sigma(\overset{\circ}{t}_0, t_1)} \kappa \Phi(\overset{\circ}{\gamma}(\overset{\circ}{t}_0)) \\ & \left. - \int_{t=\overset{\circ}{t}_0}^{t=t_1} \frac{\Sigma(\overset{\circ}{t}_0, t)}{2 S(t)} \kappa \delta \rho(\overset{\circ}{\gamma}(t)) dt + \frac{1}{\Sigma(\overset{\circ}{t}_0, t_1)} \int_{t=\overset{\circ}{t}_0}^{t=t_1} \frac{\Sigma^2(\overset{\circ}{t}_0, t)}{2 S(t)} \kappa \delta \rho(\overset{\circ}{\gamma}(t)) dt \right\} + O(\kappa^2) \end{aligned} \quad (6.21)$$

by successive integration by parts and the use of (5.11), (5.12), (5.16), (5.17) and (6.14). Again $\overset{\circ}{t}_0$ is given in terms of z_1 and t_1 by means of (3.1). Equation (6.21) is the correction to the apparent magnitude redshift relation for Newtonian perturbations of FRW cosmologies subject to the gauge fixing condition (6.14).

7. THE AVERAGING PROCEDURE

Suppose Φ is smooth on each level surface of t . Let ${}_t$ be a 3-domain in one such surface. By definition, the mean mass density perturbation on ${}_t$ is

$$\langle \delta \rho \rangle_t := \frac{\int_t \delta \rho d\overset{\circ}{v}}{\int_t d\overset{\circ}{v}} \quad (7.1)$$

where $d\overset{\circ}{v}$ is the elemental 3-volume on ${}_t$ with respect to $\overset{\circ}{\mathbf{h}}$. By means of (7.1) and (5.18) one obtains

$$\langle \delta \rho \rangle_t = -\frac{3k}{S^3(t)} \langle \Phi \rangle_t - \frac{\int_{\partial_t} {}^{(3)}\nabla_{\overset{\circ}{\mathbf{n}}} \Phi d\overset{\circ}{v}}{S^3(t) \int_t d\overset{\circ}{v}} + O(\kappa) \quad (7.2)$$

by means of the divergence theorem, where $\overset{\circ}{\mathbf{n}}$ is the unit outward pointing normal to ${}_t$ at ∂_t with respect to $\overset{\circ}{\mathbf{h}}$ in the surface $\{t = \text{const.}\}$, and where

$$\langle \Phi \rangle_t := \frac{\int_t \Phi d\overset{\circ}{v}}{\int_t d\overset{\circ}{v}} \quad (7.3)$$

is the mean value of Φ on ${}_t$. Note that $\langle \Phi \rangle_t$ is independent of t . It will be assumed that the perturbed matter distribution is sufficiently uniform on the large scale that, in the limit ${}_t \rightarrow \infty$, the surface integral term in (7.2) tends to zero, whilst $\langle \delta \rho \rangle_t$ and $\langle \Phi \rangle_t$ tend to limits $\langle \delta \rho \rangle$ and $\langle \Phi \rangle$ respectively. Under these conditions one obtains

$$\langle \delta \rho \rangle_t = -\frac{3k}{S^3(t)} \langle \Phi \rangle + O(\kappa) . \quad (7.4)$$

In terms of the dimensionless density parameter Ω of (5.2) this gives

$$\langle \delta \Omega \rangle_t = \frac{\kappa \langle \delta \rho \rangle_t}{3\overset{\circ}{H}^2(t)} + O(\kappa^2) = (1 + \overset{\circ}{q}(t)) \frac{\kappa \langle \Phi \rangle}{S(t)} + O(\kappa^2) \quad (7.5)$$

by means of (2.9). Equation (7.4) and the right side of (7.5) must be considered invalid for $k = 0$ since in that case $\langle \Phi \rangle$ must be infinite to give a finite $\langle \delta \rho \rangle_t$.

In order to obtain the mean correction to the apparent magnitude-redshift relation, one replaces the function $\Phi(\gamma(t))$ on the right of (6.21) with its mean value $\langle \Phi \rangle$, and $\frac{d}{dt} \Big|_{t=t_0} \Phi(\gamma(t))$ by its expectation value of zero. In practice, since all cosmological sources lie within gravitational potential wells, the $\frac{d}{dt} \Big|_{t=t_0} \Phi(\gamma(t))$ term in (6.21) will always give a positive contribution to the apparent magnitude, but this will be so small as to be negligible for present purposes. By means of (5.20),

(7.4) and (2.8), and for $\overset{\circ}{t}_0$ given in terms of t_1 and z_1 by (3.1), one thus obtains

$$\begin{aligned} \langle \delta m(z_1) \rangle = & -\frac{5}{2 \ln 10} S^3(t_1) \overset{\circ}{H}^2(t_1) \langle \delta \Omega \rangle_{t_1} \left\{ -\frac{S^2(\overset{\circ}{t}_0)}{\dot{S}(t_1) \Sigma(\overset{\circ}{t}_0, t_1)} \int_{t=\overset{\circ}{t}_0}^{t=t_1} \dot{F}(t) dt + 3 \int_{t=\overset{\circ}{t}_0}^{t=t_1} \frac{\Sigma(\overset{\circ}{t}_0, t)}{S^4(t)} dt \right. \\ & \left. - \frac{1}{\Sigma(\overset{\circ}{t}_0, t_1)} \int_{t=\overset{\circ}{t}_0}^{t=t_1} \Sigma(\overset{\circ}{t}_0, t) \dot{F}(t) dt - \frac{3}{\Sigma(\overset{\circ}{t}_0, t_1)} \int_{t=\overset{\circ}{t}_0}^{t=t_1} \frac{\Sigma^2(\overset{\circ}{t}_0, t)}{S^4(t)} dt \right\} + O(\kappa^2) \end{aligned} \quad (7.6)$$

for the mean correction to the apparent magnitude-redshift relation for Newtonian perturbations of FRW cosmologies. This is the fundamental equation of the paper. It could alternatively have been derived directly from (4.33) with the help of the observation that the first term in the braces on the right of (6.13) gives a zero contribution to $\langle \delta m(z_1) \rangle$. For nearby sources (7.6) gives

$$\langle \delta m(z_1) \rangle = -\frac{5}{4 \ln 10} \langle \delta \Omega \rangle_{t_1} z_1 + O(z_1^2) + O(\kappa^2) \quad (7.7)$$

by means of (3.13) and (7.5).

In the particular case of a zero mean density perturbation, $\langle \delta \rho \rangle_t = 0$, equation (7.6) clearly gives that there is a zero mean correction to the apparent magnitude-redshift relation to first order in κ . In the case of a non-zero mean density perturbation $\langle \delta \rho \rangle_t \neq 0$, one may compare either with the background metric $\overset{\circ}{\mathbf{g}}$ or with that of the homogenised cosmology determined from \mathbf{g} by the constant potential $\langle \Phi \rangle$. (See §8*a*.) In the latter case the mean perturbation of the potential is zero, so the mean density perturbation is also zero. Hence the correction to the apparent magnitude-redshift relation is zero, to first order in κ .

It is worthwhile to consider the case $\langle \delta \rho \rangle_t > 0$ in more detail. Of the four terms in the braces on the right of (7.6), the first three are positive. And though the fourth term is negative, it is evidently dominated by the second. The sum of the terms in the braces is therefore positive. This shows that an object at a given redshift appears brighter than in the reference FRW model. Moreover an object at a given redshift in a high density universe appears brighter than in a low density universe. These effects can be regarded as the gravitational lensing of the universe as a whole.

A fruitful approach to clumped matter perturbations of FRW cosmologies, introduced by Dyer & Roeder (1973), is to assume that of all the matter present, a proportion α is uniformly distributed and pressure-free, while the remaining proportion $1 - \alpha$ is gravitationally bound into clumps. In order to compute the lensing of light beams that remain far from all clumps it suffices to take into consideration the effect of only the uniformly distributed matter. In particular, shear can be neglected. The angular diameter distance (Schneider et al. 1992, eq. (3.66)) determined from such clump-avoiding light beams is often called the Dyer-Roeder distance. For light beams of large angular diameter, one expects the cumulative lensing effect to approach that of the homogenised FRW cosmology. The transition between these two regimes has been studied by Linder (1998) and found to occur typically between 1 and 10 arcseconds.

The Dyer-Roeder method may be implemented in the present framework as follows. The background metric $\overset{\circ}{\mathbf{g}}$ is taken to be the metric of the homogenised FRW model. The perturbation $\delta\mathbf{h}$ then corresponds to the removal of a proportion α of the matter of this model, followed by the addition of the same quantity of clumped matter. The apparent magnitude-redshift relation for narrow, clump-avoiding light beams is then given by (7.6), with $\langle\delta\Omega\rangle_{t_1}$ replaced by $-\alpha\langle\Omega\rangle_{t_1}$. By means of (2.8) this yields

$$\begin{aligned} \langle\delta m(z_1)\rangle = & \frac{5}{2\ln 10} S^3(t_1) \overset{\circ}{H}^2(t_1) \alpha \langle\delta\Omega\rangle_{t_1} \left\{ -\frac{S^2(\overset{\circ}{t}_0)}{\dot{S}(t_1)\Sigma(\overset{\circ}{t}_0, t_1)} \int_{t=\overset{\circ}{t}_0}^{t=t_1} \dot{F}(t) dt + 3 \int_{t=\overset{\circ}{t}_0}^{t=t_1} \frac{\Sigma(\overset{\circ}{t}_0, t)}{S^4(t)} dt \right. \\ & \left. - \frac{1}{\Sigma(\overset{\circ}{t}_0, t_1)} \int_{t=\overset{\circ}{t}_0}^{t=t_1} \Sigma(\overset{\circ}{t}_0, t) \dot{F}(t) dt - \frac{3}{\Sigma(\overset{\circ}{t}_0, t_1)} \int_{t=\overset{\circ}{t}_0}^{t=t_1} \frac{\Sigma^2(\overset{\circ}{t}_0, t)}{S^4(t)} dt \right\} + O(\kappa^2) . \end{aligned} \quad (7.8)$$

Since the expression in the braces is positive for $z_1 > 0$, this correction is positive for $z_1 > 0$. Sources viewed at a given redshift along clump-avoiding light beams thus appear dimmer than for an all-sky average. This is as one would expect since clump-avoiding light beams are less focussed. For larger angular scales, the mean correction is no longer given by (7.8) and should be expected to tend to zero since the mean density perturbation is zero. Thus (7.8) may be interpreted as the mean correction to the apparent magnitude-redshift relation for narrow, clump-avoiding light beams relative to that for wide angle beams.

The standard implementation of the Dyer-Roeder ansatz (e.g. Schneider et al. 1992, p.138 et seq.) gives the Dyer-Roeder distance as a solution to the Dyer-Roeder equation which describes light propagation through a uniformly underdense region of space-time. From the Dyer-Roeder distance one can obtain an apparent magnitude-redshift relation. On the other hand, an apparent magnitude-redshift relation for light propagation through a uniformly underdense region of space-time is also described by (7.8) for an appropriately valued constant $\langle\Phi\rangle$ (see 8.8). Nonetheless one cannot expect these two relations to agree unless similar gauge fixing conditions are applied in each case (see (8.3)).

8. EXAMPLES

Example 8.1 (Uniform density perturbations). In the special case

$$\Phi = \text{const.} \quad (8.1)$$

the perturbation (5.5) reduces to

$$\delta h_{ij} = \kappa G(t) \Phi \overset{\circ}{h}_{ij} + O(\kappa^2) . \quad (8.2)$$

To first order in κ this is equivalent to leaving the 3-metric $\overset{\circ}{\mathbf{h}}$ in (2.1) fixed and perturbing the scale factor $S(t)$ according to $S(t) \rightarrow (1 + \frac{1}{2}G(t)\kappa\Phi)S(t)$. The coordinate freedom (6.11), (6.12) reduces to the freedom to change t by an additive constant. One sees directly from (8.2) that the gauge condition (4.36) is satisfied iff $G(t)$ satisfies

$$\dot{G}(t_1) = 0 . \quad (8.3)$$

From (5.8), (5.9) and (5.10) one has that the perturbation is pressure-free to first order in κ , and so of a form as discussed in §5, if $G(t)$ satisfies

$$kG(t) = \frac{1}{S(t)}(S^3(t)\dot{G}(t))' + O(\kappa) . \quad (8.4)$$

If $G(t)$ were to vanish at $t = t_1$ then, by (8.3) and (8.4), $G(t)$ would vanish for all t . For consistency with (6.16) one may choose

$$G(t_1) = \frac{1}{S(t_1)} . \quad (8.5)$$

By (5.18) and (5.19) one has

$$\delta\rho = -\frac{3k}{S^3(t)}\Phi + O(\kappa) \quad (8.6)$$

$$\delta p = 0 + O(\kappa) . \quad (8.7)$$

By (7.5) the perturbation of the dimensionless density parameter is given by

$$\delta\Omega(t) = \frac{1}{2}(1 + q(t))\frac{\kappa\Phi}{S(t)} + O(\kappa^2) . \quad (8.8)$$

Example 8.2 (Point particle perturbations). A perturbation by the introduction of a family of comoving point particles with world lines $\{\mathbf{x} = \mathbf{x}_\alpha\}$, $\alpha = 1, 2, \dots$ and masses $m_\alpha > 0$ is described by a density perturbation of the form

$$\delta\rho(t, \mathbf{x}) = \frac{1}{S^3(t)} \sum_{\alpha} m_{\alpha} {}^{(3)}\overset{\circ}{\delta}_{\mathbf{x}_{\alpha}}(\mathbf{x}) \quad (8.9)$$

where ${}^{(3)}\overset{\circ}{\delta}_{\mathbf{x}_{\alpha}}(\mathbf{x})$ is the Dirac distribution with respect to $\overset{\circ}{\mathbf{h}}$, centred on \mathbf{x}_{α} , on each level surface of t . The particles shall represent the galaxies.

In order to satisfy (8.9) and (5.18) one seeks a potential $\Phi(\mathbf{x})$ satisfying

$$-({}^{(3)}\overset{\circ}{\Delta}\Phi(\mathbf{x}) + 3k\Phi(\mathbf{x})) = \sum_{\alpha} m_{\alpha} {}^{(3)}\overset{\circ}{\delta}_{\mathbf{x}_{\alpha}}(\mathbf{x}) . \quad (8.10)$$

Since this equation is linear one may decompose $\Phi(\mathbf{x})$ as a sum

$$\Phi(\mathbf{x}) = \sum_{\alpha} \Phi_{\alpha}(\mathbf{x}) . \quad (8.11)$$

In order to consider a typical summand $\Phi_{\alpha}(\mathbf{x})$ it is convenient to express the 3-metric $\overset{\circ}{\mathbf{h}}$ in the form

$$\overset{\circ}{\mathbf{h}} = \begin{cases} \frac{1}{k}(d\omega_{\alpha}^2 + \sin^2 \omega_{\alpha} d\Omega_{\alpha}^2) & \text{if } k > 0 \\ dr_{\alpha}^2 + r_{\alpha}^2 d\Omega_{\alpha}^2 & \text{if } k = 0 \\ \frac{1}{(-k)}(d\omega_{\alpha}^2 + \sinh^2 \omega_{\alpha} d\Omega_{\alpha}^2) & \text{if } k < 0 \end{cases} \quad (8.12)$$

where $d\Omega_{\alpha}^2$ is the 2-sphere metric and the radial coordinate ω_{α} is defined by

$$\omega_{\alpha}(r_{\alpha}) := \begin{cases} 2 \tan^{-1} \left(\frac{\sqrt{k}}{2} r_{\alpha} \right) & \text{if } k > 0 \\ 2 \tanh^{-1} \left(\frac{\sqrt{-k}}{2} r_{\alpha} \right) & \text{if } k < 0 \end{cases} \quad (8.13)$$

with the origin $\omega_{\alpha} = 0$ being the point \mathbf{x}_{α} of (8.9). The range of ω_{α} is $0 \leq \omega_{\alpha} \leq \pi$ if $k > 0$ and $0 \leq \omega_{\alpha} < \infty$ if $k < 0$.

The general radial solution to (8.10), as found by Newman & McVittie (1982), is

$$\Phi_\alpha(\mathbf{x}) = \begin{cases} \frac{\sqrt{k}}{4\pi \sin \omega_\alpha} (m_\alpha \cos 2\omega_\alpha + C_\alpha \sin 2\omega_\alpha) & \text{if } k > 0 \\ \frac{m_\alpha}{4\pi r} + C_\alpha & \text{if } k = 0 \\ \frac{\sqrt{-k}}{4\pi \sinh \omega_\alpha} (m_\alpha \cosh 2\omega_\alpha + C_\alpha \sinh 2\omega_\alpha) & \text{if } k < 0 \end{cases} \quad (8.14)$$

where C_α is an arbitrary constant.

The case $k > 0$ would appear to be the simplest insofar as the level surfaces of t are compact, so physical plausibility demands that there are at most finitely many particles present. However it is evident from (8.14) in this case that for each particle there is a complementary particle of equal mass located at the antipodal point in each surface $\{t = \text{const.}\}$. This bizarre doppelgänger phenomenon leads one to question whether the $k > 0$ solution is realistic after all. It is unclear whether the problem is an artifact of the symmetry of the level surfaces of t or of the linear approximation. There may even be a deeper issue here concerning the constraint components of the Einstein equations (D'Eath 1976).

For $k > 0$ an integration of (8.10) over a level surface of t with respect to the volume element $d\overset{\circ}{v}$ associated with $\overset{\circ}{\mathbf{h}}$ yields

$$-3k \int_{t=\text{const.}} \Phi d\overset{\circ}{v} = \sum_{\alpha} m_{\alpha} . \quad (8.15)$$

Setting $^{(3)}\overset{\circ}{V} := \int_{t=\text{const.}} d\overset{\circ}{v}$ one thus obtains

$$\langle \delta\rho \rangle_t := \frac{\sum_{\alpha} m_{\alpha}}{S^3(t) ^{(3)}\overset{\circ}{V}} = -\frac{3k}{S^3(t)} \langle \Phi \rangle \quad (8.16)$$

for

$$\langle \Phi \rangle = \frac{\int_{t=\text{const.}} \Phi d\overset{\circ}{v}}{\int_{t=\text{const.}} d\overset{\circ}{v}} . \quad (8.17)$$

Note that (8.16) agrees with (7.4) even though Φ is not smooth in the present case.

In the case $k < 0$ case the surfaces $\{t = \text{const.}\}$ have infinite volume, so infinitely many particles (or none) are needed in order to achieve a distribution that is uniform on the large scale. For each α one must choose $C_\alpha = -m_\alpha$ in order that Φ_α decays to zero at infinity. Each $\Phi_\alpha(\mathbf{x})$ is then integrable on the level surfaces $\{t = \text{const.}\}$ and a simple reciprocity argument indicates that for a sufficiently uniform distribution of particles $\Phi(\mathbf{x}) = \sum_{\alpha} \Phi_\alpha(\mathbf{x})$ should converge everywhere other than on the world lines of the particles. Let ${}_t$ be a compact 3-domain in a level surface of t . An integration of (8.10) over ${}_t$ with respect to $d\overset{\circ}{v}$ yields

$$-3k \int_{{}_t} \Phi d\overset{\circ}{v} = \sum_{\alpha} m_{\alpha} + \int_{\partial_t} \overset{\circ}{\nabla}_{\overset{\circ}{\mathbf{n}}} \Phi d\overset{\circ}{a} \quad (8.18)$$

where $\overset{\circ}{\mathbf{n}}$ is the outward unit normal at ∂_t , $d\overset{\circ}{a}$ is the area element on ∂_t induced by $\overset{\circ}{\mathbf{h}}$, and where, in the first term on the right, the sum is carried over all α such that

$\mathbf{x}_\alpha \in t$. Setting $\overset{\circ}{V}(t) := \int_t \overset{\circ}{d}v$ one obtains

$$\langle \delta\rho \rangle_t = -\frac{3k}{S^3(t)} \langle \Phi \rangle_t - \frac{\int_t \overset{\circ}{\nabla}_{\mathbf{n}} \Phi \, d\overset{\circ}{a}}{S^3(t) \overset{(3)}{\overset{\circ}{V}}(t)} \quad (8.19)$$

where

$$\langle \delta\rho \rangle_t := \frac{\sum_\alpha m_\alpha}{S^3(t) \overset{(3)}{\overset{\circ}{V}}(t)} \quad (8.20)$$

is the mean density perturbation on t and

$$\langle \Phi \rangle_t := \frac{\int_t \Phi \, d\overset{\circ}{v}}{\overset{(3)}{\overset{\circ}{V}}(t)} \quad (8.21)$$

is the mean value of Φ on t . Reasonable uniformity conditions on the distribution of the particles should ensure that $\langle \delta\rho \rangle_t$ and $\langle \Phi \rangle_t$ tend to limits $\langle \delta\rho \rangle_t$ and $\langle \Phi \rangle_t$ respectively for arbitrarily large t . Such conditions should also ensure that the second term on the right of (8.19) tends to zero for large t . One then obtains (7.4) for $k < 0$, again even though Φ is not smooth.

In the $k = 0$ case $\Phi_\alpha(r_\alpha)$, as given by (8.14), decays to zero at infinity only if $C_\alpha = 0$, and then only as $1/r_\alpha$. For a uniform distribution of particles the potential $\Phi = \sum_\alpha \Phi_\alpha$ would be infinite everywhere, so the theory breaks down in this case.

For $k \neq 0$ the perturbation of the space-time metric corresponding to the introduction of the uniform distribution of comoving point particles is described by (2.13), (5.5), (8.11) and (8.14), and the mean correction to the apparent magnitude-redshift relation, relative to the background metric, is given by (7.6).

Example 8.3 (Swiss cheese model). This model, proposed by Einstein & Strauss (1945,1946), is an exact C^{1-} solution to the Einstein equations consisting of a pressure-free FRW model in which spherical regions are replaced by spherical pieces of Schwarzschild geometry. The idea is that each of these spherical ‘holes’ represents the condensation of dust into a star, represented by the singularity at the centre. For present purposes, the singularities will be considered to represent the galaxies and the intervening dust, the ‘cheese’, will represent the intergalactic medium. The model will be considered here in terms of the perturbative formalism of §5.

Let $\rho(t)$ be the density of the intergalactic dust. In order to describe a spherically symmetric hole with a central point particle of mass m and coordinate radius $\omega = \hat{\omega}$ if $k \neq 0$, or $r = \hat{r}$ if $k = 0$, one may seek a radial potential function $\Phi(\mathbf{x})$ which satisfies

$$-(\overset{(3)}{\Delta}\Phi(\mathbf{x}) + 3k\Phi(\mathbf{x})) = -S^3(t_1)\rho(t_1) + m \overset{(3)}{\delta}_0(\mathbf{x}) \quad (8.22)$$

in the hole and matches in a C^{2-} manner to a constant potential $\Phi(\mathbf{x}) = \hat{\Phi}$ outside the hole. The first term on the right of (8.22) ensures that the hole is a vacuum.

The general radial solution to (8.22) has the form

$$\Phi(\mathbf{x}) = \begin{cases} \frac{\sqrt{k}m}{4\pi} \frac{\cos 2\omega}{\sin \omega} + \frac{C}{8\pi} \frac{\sin 2\omega}{\sin \omega} + \frac{S^3(t_1)\rho(t_1)}{3k} & \text{if } k > 0 \\ \frac{m}{4\pi r} + \frac{C}{4\pi} + \frac{1}{6}S^3(t_1)\rho(t_1)r^2 & \text{if } k = 0 \\ \frac{\sqrt{-k}m}{4\pi} \frac{\cosh 2\omega}{\sinh 2\omega} + \frac{C}{8\pi} \frac{\sinh 2\omega}{\sinh \omega} + \frac{S^3(t_1)\rho(t_1)}{3k} & \text{if } k < 0 \end{cases} \quad (8.23)$$

within the hole, where C is a constant.

In order for the solution (8.23) to join in a C^2 -manner to the constant solution $\Phi(\mathbf{x}) = \hat{\Phi}$ outside the hole, the radial derivative of $\Phi(\mathbf{x})$ must vanish at the boundary. By means of the divergence theorem, an integration of (8.22) over the hole therefore yields

$$m = S^3(t_1)\rho(t_1) {}^{(3)}\mathring{V}_{hole} - 3k \int_{hole} \Phi d\mathring{v} \quad (8.24)$$

for all values of k , where ${}^{(3)}\mathring{V}_{hole} := \int_{hole} d\mathring{v}$ is the volume of the hole with respect to \mathbf{h} . Thus the mean matter density of the hole is precisely $\rho(t_1)$ for $k = 0$ and $\rho(t_1) + O(\hat{\omega}^2)$ for $k \neq 0$. Substitution of (8.23) into (8.24) yields

$$C = \begin{cases} -2\sqrt{k}m \frac{(\cos \hat{\omega} - \frac{1}{3}\cos 3\hat{\omega})}{(\sin \hat{\omega} - \frac{1}{3}\sin 3\hat{\omega})} & \text{if } k > 0 \\ -2\sqrt{-k}m \frac{(\frac{1}{3}\cosh 3\hat{\omega} - \cosh \hat{\omega})}{(\frac{1}{3}\sinh 3\hat{\omega} - \sinh \hat{\omega})} & \text{if } k < 0, \end{cases} \quad (8.25)$$

whilst in the case $k = 0$ (8.24) gives

$$m = \frac{4\pi}{3}\rho(t_1)S^3(t_1)\hat{r}^3 \quad \text{if } k = 0. \quad (8.26)$$

For $k \neq 0$ the continuity of Φ at the boundary of the hole gives

$$\hat{\Phi} = \begin{cases} \frac{S^3(t_1)\rho(t_1)}{3k} - \frac{\sqrt{k}m}{3\pi} \frac{1}{(\sin \hat{\omega} - \frac{1}{3}\sin 3\hat{\omega})} & \text{if } k > 0 \\ \frac{S^3(t_1)\rho(t_1)}{3k} + \frac{\sqrt{-k}m}{3\pi} \frac{1}{(\frac{1}{3}\sinh 3\hat{\omega} - \sinh \hat{\omega})} & \text{if } k < 0 \end{cases} \quad (8.27)$$

by means of (8.23) and (8.25), whilst in the case $k = 0$ one obtains

$$\hat{\Phi} = \frac{3m}{8\pi\hat{r}} + \frac{C}{4\pi} \quad \text{if } k = 0 \quad (8.28)$$

by means of (8.23) and (8.24).

Suppose now that there are many holes, each labelled by an index α . Let t be a compact 3-domain in a level surface of t such that ∂_t intersects none of the holes. The mean value of Φ on t is

$$\langle \Phi \rangle_t := \frac{\int_t \Phi d\mathring{v}}{{}^{(3)}\mathring{V}(t)} = \hat{\Phi} + \frac{\sum_{\alpha} \int_{hole_{\alpha}} (\Phi - \hat{\Phi}) d\mathring{v}}{{}^{(3)}\mathring{V}(t)} \quad (8.29)$$

where ${}^{(3)}\mathring{V}(t) := \int_t d\mathring{v}$ is the volume of t with respect to \mathbf{h} and the sum is carried out over all α such that the α^{th} hole is contained in t . By means of (8.24) and

(8.27) in the cases $k \neq 0$, and by means of (8.23), (8.26) and (8.28) in the case $k = 0$, one has

$$\langle \Phi \rangle_t = \hat{\Phi} + M_t \quad (8.30)$$

where

$$M_t = \begin{cases} \frac{1}{(3)V(t)} \cdot \frac{1}{3} \sum_{\alpha} \frac{m_{\alpha}}{k} \left\{ \frac{2(\hat{\omega}_{\alpha} - \frac{1}{2} \sin 2\hat{\omega}_{\alpha})}{(\sin \hat{\omega}_{\alpha} - \frac{1}{3} \sin 3\hat{\omega}_{\alpha})} - 1 \right\} & \text{if } k > 0 \\ \frac{1}{(3)V(t)} \cdot \frac{1}{10} \sum_{\alpha} m_{\alpha} \hat{r}_{\alpha}^2 & \text{if } k = 0 \\ \frac{1}{(3)V(t)} \cdot \frac{1}{3} \sum_{\alpha} \frac{m_{\alpha}}{k} \left\{ 1 - \frac{2(\frac{1}{2} \sinh 2\hat{\omega}_{\alpha} - \hat{\omega}_{\alpha})}{(\frac{1}{3} \sinh 3\hat{\omega}_{\alpha} - \sinh \hat{\omega}_{\alpha})} \right\} & \text{if } k < 0 . \end{cases} \quad (8.31)$$

It will be assumed that the distribution of holes is sufficiently uniform that M_t and $\langle \Phi \rangle_t$ tend to limits M and $\langle \Phi \rangle$ respectively as t becomes arbitrarily large. In order to obtain $\langle \Phi \rangle = 0$ one must have

$$\hat{\Phi} + M = 0 . \quad (8.32)$$

By (7.4) one then has

$$\langle \delta \rho \rangle_t = 0 + O(\kappa) . \quad (8.33)$$

In the case $k = 0$ the quantities \hat{m}_{α} and \hat{r}_{α} are, for each α , related by an equation of the form (8.24), whereby one has

$$\frac{M_t}{S(t_1)} = \left(\frac{3\pi}{4\rho(t_1)} \right)^{2/3} \frac{\sum_{\alpha} m_{\alpha}^{5/3}}{S^3(t_1) (3)V(t)} \quad \text{if } k = 0 . \quad (8.34)$$

For $k \neq 0$ the quantities \hat{m}_{α} and \hat{r}_{α} are, for each α , related by an equation of the form (8.27) which involves the constant $\hat{\Phi}$. However the holes may be assumed sufficiently small that (8.34) is a valid approximation for $k \neq 0$. By (8.8) and (8.34), the dimensionless density parameter of the intergalactic matter is then to be perturbed by an amount

$$\delta \Omega_{cheese} = -(1 + q(t_1)) \frac{\kappa M}{S(t_1)} \quad \text{if } k \neq 0 \quad (8.35)$$

for

$$\frac{M}{S(t_1)} = \left(\frac{3\pi}{4\rho(t_1)} \right)^{2/3} \lim_{t_1 \rightarrow \infty} \frac{\sum_{\alpha} m_{\alpha}^{5/3}}{\overset{\circ}{V}_{t_1}(t_1)} , \quad (8.36)$$

the sum being carried over all α for which the α^{th} hole is contained in t_1 , with $\overset{\circ}{V}_{t_1}(t_1) = S^3(t_1) (3)V(t_1)$ the volume of t_1 with respect to the 3-metric $S^3(t_1)\mathbf{h}$ induced by $\overset{\circ}{\mathbf{g}}$ on $\{t = t_1\}$. If one regards the $k = 0$ case as a limit of the $k \neq 0$ cases then (8.35) may be considered to apply for all k .

It is the perturbation (8.35) of the density of the intergalactic matter that distinguishes the present approach to the Swiss cheese model from those of other authors, including Dyer & Roeder (1974). Only with such a perturbation will the mean density parameter $\langle \Omega \rangle_t$ be unperturbed. And only then, as discussed in §5, will there be a zero mean correction to the apparent magnitude-redshift relation to first order in κ .

9. CONCLUDING REMARKS

The validity of the linear approximation employed in this paper depends upon the smallness of the mean dimensionless density perturbation $\langle \delta\Omega \rangle_t$ associated with the matter in the galaxies. Astronomical estimates of this quantity are at present inconclusive, although there is a general consensus that one does have $\langle \delta\Omega \rangle_t \ll 1$. If this is correct then the theory presented should provide a valid description of the total gravitational field and cumulative weak lensing effects of the galaxies for any given model of the distribution of galactic matter. Depending upon the actual value of $\langle \delta\Omega \rangle_t$ and the range of redshifts of interest, there may be a need to carry the theory to second or higher order in κ . This would at least take into account the effect of shear on the apparent brightness of cosmological sources. Caustics however cannot be adequately described by any finite order power series analysis. For this it would be necessary to consider the full non-linear theory.

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